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Global Existence and Exponential Stability of Convection

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Consider thermal convection phenomena of a viscous incompressible fluid in a bounded domain under the influence of gravity. It is proved that when some parameters are small enough, a strong solution near a steady state exists globally in time and uniformly goes to the steady state as $t \to \infty$ with an exponential rate.

1. INTRODUCTION

The purpose of the present paper is to prove the global existence, asymptotic behavior, and stability of solutions to the interior convection problem. Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary Γ and let it be occupied by a viscous incompressible fluid. Our concern is the thermal gravitational convection of such a fluid heated at a part of the boundary $\Gamma_0 \subset \Gamma$. As a typical case, the following type domain Ω is kept in mind: Γ consists of two connected components Γ_0 and $\Gamma_1 \equiv \Gamma \backslash \Gamma_0$, and Γ_0 is contained in the domain interior to Γ_1. Since the temperature on Γ_0 is larger than that on Γ_1, the buoyancy caused by the gravitational field $g_0 \nabla \psi(x)$ derives the convective motion if the temperature gradient and the gravitational constant $g_0 > 0$ are large enough to overcome the stabilizing effect of viscosity and thermal conductivity. Here, the gravitational potential $\psi(x)$ is a smooth function on $\overline{\Omega}$ and is normalized like $\|\nabla\psi\|_\infty = 1$ ($\|\cdot\|_\infty = \max_{x\in\overline{\Omega}} |\cdot|$).

To consider the above phenomena mathematically, the model equations

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derived from the Boussinesq approximation (Chandrasekhar [2]) are usually applied. They are given by

$$\partial_t v + v\cdot\nabla v = \nu\Delta v + g_0(\nabla\psi)\{1-\eta(\theta-T_0)\} - \nabla\pi \qquad (x\in\Omega, t>0), \quad (1.1)$$

$$\nabla\cdot v = 0 \qquad (x\in\Omega, t\geq 0), \quad (1.2)$$

$$\partial_t\theta + v\cdot\nabla\theta = \kappa\Delta\theta \qquad (x\in\Omega, t>0), \quad (1.3)$$

where the velocity $v(x,t) = (v^1(x,t), v^2(x,t), v^3(x,t))$, the temperature $\theta(x,t)$ and the pressure $\pi(x,t)$ are unknowns, while the kinematic viscosity ν, the volume expansion coefficient η, and the thermal conductivity κ are positive constants. We consider (1.1)–(1.3) subject to boundary conditions

$$v=0, \qquad \theta = T_w(x) \neq \text{constant} \qquad (x\in\Gamma, t>0), \qquad (1.4)$$

where $T_w(x)$ is assumed to be a non-constant continuous function on Γ. The constant T_0 appearing in (1.1) is given by $T_0 = \min_{x\in\Gamma} T_w(x)$. Let $\varphi(x) \in C^2(\Omega)\cap C(\overline{\Omega})$ be a (classical) solution to

$$\Delta\varphi = 0\,(x\in\Omega), \qquad \varphi = T_w(x)\,(x\in\Gamma).$$

By the assumption on $T_w(x)$, the temperature gradient does not vanish; that is, $\|\nabla\varphi\|_\infty \neq 0$. Thus we set

$$T(x) = \frac{T_w(x)-T_0}{\|\nabla\varphi\|_\infty}\,(x\in\Gamma), \qquad \phi(x) = \frac{\varphi(x)-T_0}{\|\nabla\varphi\|_\infty}\,(x\in\overline{\Omega}). \qquad (1.5)$$

Then $\phi(x)$ is, of course, a harmonic function with values $T(x)$ on Γ, and with a normalized gradient $\|\nabla\phi\|_\infty = 1$.

We now make the following change of variable and functions:

$$\nu t = t^*, v = \nu v^*, \theta - T_0 = \sqrt{\frac{\nu^3\|\nabla\varphi\|_\infty}{\eta g_0\kappa}}\,\theta^*, \pi - g_0\psi = \nu^2\pi^*.$$

By omitting the asterisks for notational simplicity, (1.1)–(1.4) are reduced to

$$\left.\begin{aligned}
&\partial_t v + v\cdot\nabla v = \Delta v - \alpha(\nabla\psi)\theta - \nabla\pi && (x\in\Omega, t>0),\\
&\nabla\cdot v = 0 && (x\in\Omega, t\geq 0),\\
&\partial_t\theta + v\cdot\nabla\theta = \frac{1}{\sigma}\Delta\theta && (x\in\Omega, t>0),\\
&v=0, \theta = \frac{\alpha}{\sigma}T(x) && (x\in\Gamma, t>0),\\
&v(x,0)=a(x), \theta(x,0)=b(x) && (x\in\Omega),
\end{aligned}\right. \qquad (\mathrm{P})$$

where a and b are given initial data; α and σ are positive parameters defined by

$$\alpha = \sqrt{\frac{\eta g_0 \|\nabla \varphi\|_\infty}{\kappa \nu}}, \qquad \sigma = \frac{\nu}{\kappa}. \tag{1.6}$$

In the present paper, under a smallness condition on the parameters, we prove the existence of a global strong solution to (P) which uniformly goes to a steady solution as $t \to \infty$ with an exponential rate. In this context, the steady solution is said to be asymptotically stable. We do not assume any regularity assumptions on the initial data: $\{a, b\} \in L^3 \times L^m$ for $m > 1$. For such data the local existence of a unique strong solution has been established in [11, Theorem 1]. As is well known, the space L^3 is quite important in the Navier–Stokes theory; see Giga and Miyakawa [10] and Kozono and Ozawa [14] (for the interior problem). It should be noted that the initial temperature is allowed to belong to the larger space when we take m near 1.

Until now Morimoto [17] has proved the following interesting stability criterion: if a global weak solution $\{v, \theta\}$ has the time derivative in $L^2(0, T; H^{-1})$ for all $T > 0$ (H^{-1} denotes the dual space of H_0^1), and if a steady weak solution $\{\overline{v}, \overline{\theta}\}$ has a small L^3 norm besides a smallness condition on some parameters (which is explicitly given), then $\{v - \overline{v}, \theta - \overline{\theta}\}$ decays exponentially as $t \to \infty$ in L^2. But the existence of global weak solutions which satisfy the above-mentioned assumption has not been proved. Differently from Morimoto's result, in this paper, a global solution going toward a steady solution is actually constructed, and its large time behavior is derived in the space L^∞. As for the 2-dimensional problem, she has obtained the rigorous stability theorem in [17].

On the other hand, the convection problem in the 3-dimensional exterior domain of a sphere has been discussed by Galdi and Padula [7], Chen *et al.* [4], and the author [12]. Under the influence of gravity exactly derived by the potential $1/|x|$, purely conductive steady states do exist. The stability of such states has been studied within the framework of strong L^2 solutions in [7, 12], while within that of weak solutions in [4]. In particular, algebraic L^∞ decay of disturbance was proved by means of an energy method [12], in comparison with exponential L^∞ decay for the interior problem.

The strategy for our goal is as follows. We first find a steady solution to (P) in the space $L^3 \times L^m$, making use of the contraction mapping principle. In order to prove the global existence and large time behavior of solutions, we next establish $(L^p \times L^q) - (L^r \times L^s)$ estimates for the analytic semigroup generated by the linearized operator around the steady solution obtained above. Although this principle is rather standard, the following seem to be not only important but also new:

—We establish the estimates for various cases including $p \neq q$ and $r \neq s$, which are actually required to solve (P) with $b \in L^m$ for $m < 3$. Note that such estimates for semigroups (associated with other systems) published up to now have been given only for the case where $p = q$ and $r = s$.

—The semigroup is investigated through $(L^p \times L^q) - (L^r \times L^s)$ estimates for the resolvent of the linearized operator. Such estimates clarify how the behavior of the resolvent, with respect to the spectral parameter, depends on four exponents p, q, r, and s (Lemma 4.3).

We mention that Kozono and Ozawa [14] studied the stability in L^r for Navier–Stokes flows via the characterization of the domains of fractional powers of the linearized operator instead of $L^p - L^r$ estimates for semigroups. Recently, by means of the $L^p - L^r$ estimates, the stability of exterior Navier–Stokes flows has been also discussed by Borchers and Miyakawa [1], Chen [3], and Kozono and Ogawa [13].

This paper consists of five sections. In the next section we state our main results: Theorem 1 (existence of a steady solution), Theorem 2 (global existence and stability of solutions). Theorem 1 is proved in Section 3. In Section 4 we drive the above-mentioned key estimates for the linearized operator. The final section is devoted to the proof of Theorem 2.

2. Results

Before stating our results, we introduce notation and some definitions. All functions are real-valued and, for simplicity, we use the same symbols for denoting the spaces of scalar and vector functions. Let $L^p(\Omega)$ be the usual Lebesgue space $(1 \leq p \leq \infty)$ with norm $\|\cdot\|_p$, $W^{k,p}(\Omega)$ the L^p Sobolev space of order $k \geq 0$ (integer), and $W_0^{1,p}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. When $p \neq q$, $\|\cdot\|_{\{p,q\}}$ stands for the norm of $L^p(\Omega) \times L^q(\Omega)$. By $\|\cdot\|_{\{p,q\}\to\{r,s\}}$ (resp. $\|\cdot\|_{p\to r}$) we denote the bounded operator norm from $L^p(\Omega) \times L^q(\Omega)$ to $L^r(\Omega) \times L^s(\Omega)$ (resp. from $L^p(\Omega)$ to $L^r(\Omega)$). Let $C_{0,\sigma}^\infty(\Omega)$ be the set of all solenoidal (i.e., divergence free) vector fields with components in $C_0^\infty(\Omega)$, and $L_\sigma^p(\Omega)$ the completion of $C_{0,\sigma}^\infty(\Omega)$ in $L^p(\Omega)$ $(1 < p < \infty)$. As proved in Fujiwara and Morimoto [6], we have the Helmholtz decomposition of L^p vector fields:

$$L^p(\Omega) = L_\sigma^p(\Omega) \oplus \{\nabla\pi; \pi \in W^{1,p}(\Omega)\} \qquad \text{(direct sum)}.$$

Let $P = P_p$ be the bounded projection operator from $L^p(\Omega)$ onto $L_\sigma^p(\Omega)$ associated with this decomposition. Then the Stokes operator in $L_\sigma^p(\Omega)$, $1 < p < \infty$, is defined by

$$D(A_p) = L^p_\sigma(\Omega) \cap W^{1,p}_0(\Omega) \cap W^{2,p}(\Omega), \qquad A_p v = -P_p \Delta v.$$

We also define the Laplace operator in $L^q(\Omega)$, $1 < q < \infty$, by

$$D(B_q) = W^{1,q}_0(\Omega) \cap W^{2,q}(\Omega), \qquad B_q \theta = -\Delta \theta.$$

It is well known that the Stokes operator generates a bounded analytic semigroup $\{e^{-tA_p}\}_{t\geq 0}$ of class (C_0) on all $L^p_\sigma(\Omega)$, $1 < p < \infty$ (Giga [8], Solonnikov [18]). Generation of such a semigroup $\{e^{-tB_q}\}_{t\geq 0}$ by the Laplace operator on $L^q(\Omega)$ is a classical result. From now on we sometimes drop the subscripts p and q attached to operators if there is no confusion.

Our aim is to construct a global solution to (P) with initial data $\{a, b\}$ in $L^3_\sigma(\Omega) \times L^m(\Omega)$ for $1 < m < \infty$. To do so, it is convenient to find, as the basic flow, an appropriate steady solution within the same space as the initial data. We thus define the following class of steady solutions to (P). Let $1 < m < \infty$ and $K > 0$. We say that a pair of functions $\{\bar{v}, \bar{\theta}\}$ is a steady solution of class (m, K) to (P) if

$$\bar{v} \in D(A_3),$$

$$\bar{\theta} = \tilde{\theta} + \frac{\alpha}{\sigma}\phi, \qquad \tilde{\theta} \in D(B_m), \qquad \phi\text{: the function given by (1.5)},$$

$$\|A\bar{v}\|_3 + \|B\tilde{\theta}\|_m \leq K,$$

and $\{\bar{v}, \tilde{\theta}\}$ satisfies

$$A\bar{v} + P\left\{\alpha(\nabla\psi)\tilde{\theta} + \frac{\alpha^2}{\sigma}(\nabla\psi)\phi + \bar{v}\cdot\nabla\bar{v}\right\} = 0 \qquad \text{in } L^3_\sigma(\Omega), \tag{2.1}$$

$$\frac{1}{\sigma}B\tilde{\theta} + \frac{\alpha}{\sigma}\bar{v}\cdot\nabla\phi + \bar{v}\cdot\nabla\tilde{\theta} = 0 \qquad \text{in } L^m(\Omega). \tag{2.2}$$

Besides the parameters α and σ given by (1.6), we introduce a new parameter R defined by

$$R = \frac{\eta g_0 h|\Omega|^{1/3}}{\nu^2} = \frac{\alpha^2}{\sigma}\cdot\frac{h|\Omega|^{1/3}}{\|\nabla\varphi\|_\infty}, \tag{2.3}$$

where

$$h = \max_{x\in\Gamma} T_w(x) - T_0 > 0 \qquad \left(T_0 = \min_{x\in\Gamma} T_w(x)\right).$$

Then the existence of steady solutions of class (m, K) is presented by the following theorem.

THEOREM 1. *For each $m \in (1, \infty)$, there exists a positive constant $\overline{\delta} = \overline{\delta}(m)$ such that if*

$$\alpha < \overline{\delta}, \qquad (1+\sigma)R < \overline{\delta},$$

then (P) *has a unique steady solution $\{\overline{v}, \overline{\theta}\}$ of class (m, K_0R) with* $K_0 = 4\|P\|_{3\to 3}$.

Remark 2.1. The first existence theorem for steady solutions (in general bounded domains) was given by Morimoto [15]. She proved the existence of weak solutions together with the interior regularity property. And in [16, Theorem 2] it was also proved that the weak solutions with small L^3 norm are unique.

Next, we are concerned with the stability of the steady solution $\{\overline{v}, \overline{\theta}\}$ of class (m, K_0R) in Theorem 1 and, at the same time, with the global existence of a strong solution to (P). To these ends, we set

$$v = \overline{v} + \hat{v}, \qquad \theta = \overline{\theta} + \hat{\theta}, \qquad \pi = \overline{\pi} + \hat{\pi}$$

in (P), where $\overline{\pi}$ is the steady pressure associated with $\{\overline{v}, \overline{\theta}\}$, and $\{\hat{v}, \hat{\theta}, \hat{\pi}\}$ stands for the disturbance around $\{\overline{v}, \overline{\theta}, \overline{\pi}\}$. We now define the notion of a strong solution to (P) by taking account of the above-mentioned class of $\{a, b\}$ and $\{\overline{v}, \overline{\theta}\}$. We say that a pair of functions $\{v, \theta\}$ is a strong ($L^3 \times L^m$) solution to (P) on $[0, \infty)$ if the disturbance $\{\hat{v}, \hat{\theta}\}$ is of class

$$\hat{v} = v - \overline{v} \in C([0, \infty); L^3_\sigma(\Omega)) \cap C(0, \infty; D(A_3)) \cap C^1(0, \infty; L^3_\sigma(\Omega)),$$
$$\hat{\theta} = \theta - \overline{\theta} \in C([0, \infty); L^m(\Omega)) \cap C(0, \infty; D(B_m)) \cap C^1(0, \infty; L^m(\Omega)),$$

and satisfies

$$\frac{d\hat{v}}{dt} + A\hat{v} + P\{\alpha(\nabla\psi)\hat{\theta} + \overline{v}\cdot\nabla\hat{v} + \hat{v}\cdot\nabla\overline{v} + \hat{v}\cdot\nabla\hat{v}\} = 0 \qquad (t>0), \tag{2.4}$$

$$\frac{d\hat{\theta}}{dt} + \frac{1}{\sigma}B\hat{\theta} + \overline{v}\cdot\nabla\hat{\theta} + \hat{v}\cdot\nabla\overline{\theta} + \hat{v}\cdot\nabla\hat{\theta} = 0 \qquad (t>0), \tag{2.5}$$

$$\hat{v}(0) = a - \overline{v}, \qquad \hat{\theta}(0) = b - \overline{\theta}. \tag{2.6}$$

(Also, we often say that $\{\hat{v}, \hat{\theta}\}$ itself is a strong solution to (2.4)–(2.6) on $[0, \infty)$.)

Our main theorem on the stability of the steady solution reads as follows.

THEOREM 2. *Suppose that*

$$\{a, b\} \in L^3_\sigma(\Omega) \times L^m(\Omega), \qquad 1 < m < \infty.$$

Then there exists a constant $\delta = \delta(m) \in (0, \bar{\delta}]$ *such that if*

$$\left(\frac{1}{\sigma} + \sigma\right)\alpha < \delta, \qquad (1 + \sigma)R < \delta,$$

then the steady solution $\{\bar{v}, \bar{\theta}\}$ *in Theorem* 1 *is stable in the following sense: there exists a positive constant* $\varepsilon = \varepsilon(\alpha, \sigma, R; m)$ *such that if*

$$\|a - \bar{v}\|_3 + \|b - \bar{\theta}\|_m < \varepsilon,$$

then (P) *has a unique strong solution* $\{v, \theta\}$ *on* $[0, \infty)$, *which decays like*

$$\|v(t) - \bar{v}\|_\infty + \|\theta(t) - \bar{\theta}\|_\infty = o(e^{-\mu t}),$$

as $t \to \infty$ *for some positive number* $\mu = \mu(\alpha, \sigma, R; m)$.

3. STEADY SOLUTIONS

In this section we construct a steady solution $\{\bar{v}, \bar{\theta}\}$ of class (m, K) to prove Theorem 1. We define the mapping F: $D(A_3) \times D(B_m) \to D(A_3) \times D(B_m)$ by

$$F\begin{bmatrix} \bar{v} \\ \tilde{\theta} \end{bmatrix} = \begin{bmatrix} -A^{-1}P\{\alpha(\nabla\psi)\tilde{\theta} + \dfrac{\alpha^2}{\sigma}(\nabla\psi)\phi + \bar{v}\cdot\nabla\bar{v}\} \\ -B^{-1}(\alpha\bar{v}\cdot\nabla\phi + \sigma\bar{v}\cdot\nabla\tilde{\theta}) \end{bmatrix}. \tag{3.1}$$

The (2.1) and (2.2) are equivalent to

$$\begin{bmatrix} \bar{v} \\ \tilde{\theta} \end{bmatrix} = F\begin{bmatrix} \bar{v} \\ \tilde{\theta} \end{bmatrix} \qquad \text{in } D(A_3) \times D(B_m). \tag{3.2}$$

Theorem 1 is now a direct consequence of the following proposition.

PROPOSITION 3.1. *For each $m \in (1, \infty)$, there exists a positive constant $\bar{\delta} = \bar{\delta}(m)$ such that if*

$$\alpha < \bar{\delta}, \qquad (1+\sigma)R < \bar{\delta},$$

Then F is a contraction mapping on the complete metric space

$$E_K = \{\{\bar{v}, \tilde{\theta}\} \in D(A_3) \times D(B_m); \|A\bar{v}\|_3 + \|B\tilde{\theta}\|_m \leq K\}$$

with a constant $K < 4C_0R$, where $C_0 = \|P\|_{3\to3}$.

To prove Proposition 3.1, we give the following elementary lemma which is verified with use of the Hölder inequality and the Sobolev embedding relations due to Giga [9] (cf. Giga and Miyakawa [10, Lemma 2.2], Hishida [11, Lemma 3.3]).

LEMMA 3.2. *Let $1 < p, q, r < \infty$ satisfy*

$$\frac{1}{r} > \frac{1}{p} - \frac{2}{3}, \qquad \frac{1}{r} > \frac{1}{q} - \frac{1}{3}, \qquad \frac{1}{r} \geq \frac{1}{p} + \frac{1}{q} - 1. \tag{3.3}$$

Then the estimates

$$\|P(v \cdot \nabla w)\|_r \leq C\|Av\|_p\|Aw\|_q, \tag{3.4}$$

$$\|v \cdot \nabla\theta\|_r \leq C\|Av\|_p\|B\theta\|_q, \tag{3.5}$$

hold with $C = C(p, q, r)$ independent of $v \in D(A_p)$, $w \in D(A_q)$ and $\theta \in D(B_q)$.

Proof of Proposition 3.1. In the proof, we write $\{v, \theta\}$ instead of $\{\bar{v}, \tilde{\theta}\}$. Since ϕ is given by (1.5), and since $\|\nabla\psi\|_\infty = 1$, we obtain by the maximum principle

$$\begin{aligned}\frac{\alpha^2}{\sigma}\|P\{(\nabla\psi)\phi\}\|_3 &\leq C_0\frac{\alpha^2}{\sigma}\|\phi\|_3 = C_0\frac{\alpha^2\|\varphi - T_0\|_3}{\sigma\|\nabla\varphi\|_\infty} \\ &\leq C_0\frac{\alpha^2\|\varphi - T_0\|_\infty|\Omega|^{1/3}}{\sigma\|\nabla\varphi\|_\infty} = C_0\frac{\alpha^2 h|\Omega|^{1/3}}{\sigma\|\nabla\varphi\|_\infty} = C_0R.\end{aligned} \tag{3.6}$$

Estimates (3.4), (3.6), and the embedding relation $D(B_m) \subset L^3(\Omega)$ imply

$$\left\|P\left\{\alpha(\nabla\psi)\theta + \frac{\alpha^2}{\sigma}(\nabla\psi)\phi + v \cdot \nabla v\right\}\right\|_3 \leq C_1\alpha\|B\theta\|_m + C_0R + C_2\|Av\|_3^2. \tag{3.7}$$

Similarly, (3.5) and the relation $D(A_3) \subset L^m(\Omega)$ yield

$$\|\alpha v \cdot \nabla \phi + \sigma v \cdot \nabla \theta\|_m \le C_3 \alpha \|Av\|_3 + C_4 \sigma \|Av\|_3 \|B\theta\|_m. \tag{3.8}$$

Therefore,

$$\left\| F\begin{bmatrix} v \\ \theta \end{bmatrix} \right\|_{D(A_3)\times D(B_m)} \le C_0 R + C_5 \alpha (\|Av\|_3 + \|B\theta\|_m) + C_6(1+\sigma)(\|Av\|_3 + \|B\theta\|_m)^2,$$

holds with $C_5 = C_5(m) = \max\{C_1, C_3\}$ and $C_6 = C_6(m) = \max\{C_2, C_4\}$. Set

$$\bar{\delta} = \bar{\delta}(m) = \min\left\{\frac{1}{2C_5}, \frac{1}{16 C_0 C_6}\right\},$$

$$K = \frac{1 - C_5\alpha - \sqrt{D}}{2C_6(1+\sigma)},$$

where

$$D = (1 - C_5\alpha)^2 - 4C_0C_6(1+\sigma)R.$$

If

$$\alpha < \bar{\delta} \le \frac{1}{2C_5}, \qquad (1+\sigma)R < \bar{\delta} \le \frac{1}{16C_0C_6}, \tag{3.9}$$

then we get

$$D > \frac{1}{4} - 4C_0C_6(1+\sigma)R > 0, \quad 0 < K = \frac{2C_0R}{1 - C_5\alpha + \sqrt{D}} < 4C_0R, \tag{3.10}$$

and, further,

$$F\begin{bmatrix} v \\ \theta \end{bmatrix} \in E_K \qquad \text{for all} \begin{bmatrix} v \\ \theta \end{bmatrix} \in E_K.$$

By the same way as in (3.7) and (3.8), it is seen that

$$\left\| F\begin{bmatrix} v_1 \\ \theta_1 \end{bmatrix} - F\begin{bmatrix} v_2 \\ \theta_2 \end{bmatrix} \right\|_{D(A_3)\times D(B_m)} \le \{C_5\alpha + C_6(2+\sigma)K\}\{\|A(v_1 - v_2)\|_3 + \|B(\theta_1 - \theta_2)\|_m\},$$

for $\{v_1, \theta_1\}, \{v_2, \theta_2\} \in E_K$. By (3.9) and (3.10), we have

$$C_5\alpha + C_6(2+\sigma)K < C_5\alpha + 8C_0C_6(1+\sigma)R < 1,$$

and, therefore, F is contractive on E_K. ∎

4. Linearized Operator

Let $\{\bar{v}, \bar{\theta}\}$ $(\bar{\theta} = \tilde{\theta} + (\alpha/\sigma)\phi)$ be the steady solution of class (m, K_0R) in Theorem 1. In this section we carry out the analysis of the linearized operator $L = L_{p,q}$ around $\{\bar{v}, \bar{\theta}\}$, which is given by

$$D(L_{p,q}) = D(A_p) \times D(B_q),$$
$$L = L_0 + L_1,$$
$$L_0 \begin{bmatrix} v \\ \theta \end{bmatrix} = \begin{bmatrix} Av \\ \dfrac{1}{\sigma} B\theta \end{bmatrix},$$
$$L_1 \begin{bmatrix} v \\ \theta \end{bmatrix} = \begin{bmatrix} P\{\alpha(\nabla\psi)\theta + \bar{v}\cdot\nabla v + v\cdot\nabla\bar{v}\} \\ \bar{v}\cdot\nabla\theta + v\cdot\nabla\bar{\theta} \end{bmatrix},$$

where $\{p, q\}$ is a suitable pair determined by (4.1) below. We often drop the subscripts p and q attached to L.

We begin with the following lemma concerning an estimate of L_1.

LEMMA 4.1. *Suppose that $\{p, q\}$ satisfies*

$$\begin{aligned} &-\frac{2}{3} \le \frac{1}{p} - \frac{1}{q} < \frac{2}{3}, \qquad \frac{1}{p} - \frac{1}{q} \le 1 - \frac{1}{m}, \\ &\frac{1}{q} > \frac{1}{m} - \frac{1}{3}, \qquad 1 < p, q < \infty \end{aligned} \tag{4.1}$$

Then

$$\begin{aligned} \left\| L_1 \begin{bmatrix} v \\ \theta \end{bmatrix} \right\|_{\{p,q\}} \le C_{p,q} \Bigg\{ &\left(\frac{\alpha}{\sigma} + \|A\bar{v}\|_3 + \|B\tilde{\theta}\|_m \right) \|Av\|_p \\ &+ (\alpha + \|A\bar{v}\|_3) \|B\theta\|_q \Bigg\}, \end{aligned} \tag{4.2}$$

holds for $\{v, \theta\} \in D(A_p) \times D(B_q)$ with a constant $C_{p,q} = C_{p,q}(m)$.

Proof. This lemma follows from Lemma 3.2 together with the embedding relations $D(B_q) \subset L^p(\Omega)$, $D(A_p) \subset L^q(\Omega)$. ▌

For $0 < \omega < \pi/2$, set

$$\Sigma_\omega = \{\lambda \in \mathbb{C}; |\arg \lambda| \leq \pi - \omega\} \cup \{0\}.$$

Then we have

LEMMA 4.2. *For each $\{p, q\}$ satisfying* (4.1) *and each $\omega \in (0, \pi/2)$, there is a constant $M_{p,q}(m, \omega)$ such that*

$$\left\| L_1(\lambda + L_0)^{-1} \begin{bmatrix} v \\ \theta \end{bmatrix} \right\|_{\{p,q\}} \leq M_{p,q}(m, \omega) \left\{ \left(\frac{1}{\sigma} + \sigma \right) \alpha + K_0(1 + \sigma) R \right\} \left\| \begin{bmatrix} v \\ \theta \end{bmatrix} \right\|_{\{p,q\}}, \tag{4.3}$$

for all $\lambda \in \Sigma_\omega$ and $\{v, \theta\} \in L^p_\sigma(\Omega) \times L^q(\Omega)$, where R is the parameter given by (2.3), *and K_0 is the constant in Theorem* 1.

Proof. It is well known (Giga [8]) that for each $\omega \in (0, \pi/2)$,

$$\Sigma_\omega \subset \rho(-A_p) \cap \rho(-B_q)$$

with bounds

$$\begin{aligned} K_p(\omega) &= \sup_{\lambda \in \Sigma_\omega} (1 + |\lambda|) \|(\lambda + A)^{-1}\|_{p \to p} < \infty, \\ \hat{K}_q(\omega) &= \sup_{\lambda \in \Sigma_\omega} (1 + |\lambda|) \|(\lambda + B)^{-1}\|_{q \to q} < \infty. \end{aligned} \tag{4.4}$$

In view of (4.2), we obtain by (4.4)

$$\begin{aligned} \left\| L_1(\lambda + L_0)^{-1} \begin{bmatrix} v \\ \theta \end{bmatrix} \right\|_{\{p,q\}} \leq C_{p,q} \Bigg\{ &\left(\frac{\alpha}{\sigma} + \|A\bar{v}\|_3 + \|B\bar{\theta}\|_m \right) (1 + K_p(\omega)) \|v\|_p \\ &+ (\alpha + \|A\bar{v}\|_3) \sigma (1 + \hat{K}_q(\omega)) \|\theta\|_q \Bigg\}. \end{aligned}$$

Since $\{\bar{v}, \bar{\theta}\}$ is of class $(m, K_0 R)$, we have

$$\|A\tilde{v}\|_3 + \|B\tilde{\theta}\|_m \le K_0 R.$$

Thus we are led to (4.3) with $M_{p,q}(m, \omega) = C_{p,q}(m)(1 + K_p(\omega) + \hat{K}_q(\omega))$. ∎

We next derive the key lemma on the $(L^p \times L^q) - (L^r \times L^s)$ estimates of the resolvent of $-L$.

LEMMA 4.3. *For each* $\{p, q\}$ *satisfying* (4.1) *and each* $\omega \in (0, \pi/2)$, *set*

$$\delta_{p,q} = \delta_{p,q}(m, \omega) \equiv \min\left\{\frac{1}{(1 + K_0)M_{p,q}(m, \omega)}, \bar{\delta}(m)\right\}, \tag{4.5}$$

where $\bar{\delta}(m)$ *and* $M_{p,q}(m, \omega)$ *are, respectively, the constants in Theorem* 1 *and Lemma* 4.2. *If*

$$\left(\frac{1}{\sigma} + \sigma\right)\alpha < \delta_{p,q}, \qquad (1 + \sigma)R < \delta_{p,q}, \tag{4.6}$$

then

$$\Sigma_\omega \subset \rho(-L_{p,q})$$

holds true. Furthermore, for each $j = 0, 1$, *if* $\{r, s\}$ *fulfills*

$$\begin{aligned} 0 \le \beta &\equiv \frac{3}{2}\left(\frac{1}{p} - \frac{1}{r}\right) \le 1 - \frac{j}{2}, \\ 0 \le \gamma &\equiv \frac{3}{2}\left(\frac{1}{q} - \frac{1}{s}\right) \le 1 - \frac{j}{2}, \end{aligned} \tag{4.7}$$

then there is a constant $C = C(p, q, r, s; \alpha, \sigma, R; m, \omega)$ *so that*

$$\left\|\nabla^j(\lambda + L)^{-1}\begin{bmatrix} v \\ \theta \end{bmatrix}\right\|_{\{r,s\}} \le C\{(1 + |\lambda|)^{-1+\beta+j/2} + (1 + |\lambda|)^{-1+\gamma+j/2}\}\left\|\begin{bmatrix} v \\ \theta \end{bmatrix}\right\|_{\{p,q\}}, \tag{4.8; j}$$

for all $\lambda \in \Sigma_\omega$ *and* $\{v, \theta\} \in L^p_\sigma(\Omega) \times L^q(\Omega)$.

Proof. Since (4.6) implies

$$k_{p,q} = k_{p,q}(\alpha, \sigma, R; m, \omega) \equiv M_{p,q}(m, \omega)\left\{\left(\frac{1}{\sigma} + \sigma\right)\alpha + K_0(1 + \sigma)R\right\} < 1,$$

it follows from (4.3) that for all $\lambda \in \Sigma_\omega$, a bounded inverse operator

$$[1 + L_1(\lambda + L_0)^{-1}]^{-1} = \sum_{n=0}^{\infty} \{-L_1(\lambda + L_0)^{-1}\}^n$$

exists on $L^p_\sigma(\Omega) \times L^q(\Omega)$ with bounds

$$\|[1 + L_1(\lambda + L_0)^{-1}]^{-1}\|_{\{p,q\}\to\{p,q\}} \le \frac{1}{1 - k_{p,q}}. \tag{4.9}$$

Therefore,

$$(\lambda + L)^{-1} = (\lambda + L_0)^{-1}[1 + L_1(\lambda + L_0)^{-1}]^{-1} \qquad (\lambda \in \Sigma_\omega) \tag{4.10}$$

also exists as a bounded operator on $L^p_\sigma(\Omega) \times L^q(\Omega)$; we thus obtain $\Sigma_\omega \subset \rho(-L_{p,q})$.

We next prove (4.8; j). By (4.7) we have

$$D(A_p^{\beta+j/2}) \subset W^{j,r}(\Omega), \qquad D(B_q^{\gamma+j/2}) \subset W^{j,s}(\Omega).$$

These embedding relations together with the momentum inequality and (4.4) imply that

$$\begin{aligned}
\left\|\nabla^j(\lambda + L_0)^{-1}\begin{bmatrix} v \\ \theta \end{bmatrix}\right\|_{\{r,s\}} &= \|\nabla^j(\lambda + A)^{-1}v\|_r + \left\|\nabla^j\left(\lambda + \frac{1}{\sigma}B\right)^{-1}\theta\right\|_s \\
&\le C\|A^{\beta+j/2}(\lambda + A)^{-1}v\|_p + C\left\|B^{\gamma+j/2}\left(\lambda + \frac{1}{\sigma}B\right)^{-1}\theta\right\|_q \\
&\le C\|A(\lambda + A)^{-1}v\|_p^{\beta+j/2}\|(\lambda + A)^{-1}v\|_p^{1-\beta-j/2} \\
&\quad + C\left\|B\left(\lambda + \frac{1}{\sigma}B\right)^{-1}\theta\right\|_q^{\gamma+j/2}\left\|\left(\lambda + \frac{1}{\sigma}B\right)^{-1}\theta\right\|_q^{1-\gamma-j/2} \\
&\le C(1 + K_p(\omega))(1 + |\lambda|)^{-1+\beta+j/2}\|v\|_p \\
&\quad + C\sigma(1 + \hat{K}_q(\omega))(1 + \sigma|\lambda|)^{-1+\gamma+j/2}\|\theta\|_q,
\end{aligned} \tag{4.11; j}$$

for $\lambda \in \Sigma_\omega$ and $\{v, \theta\} \in L^p_\sigma(\Omega) \times L^q(\Omega)$. Hence, in view of (4.10), combining (4.11; j) with (4.9) leads us to (4.8; j). ∎

In what follows we will fix $\omega \in (0, \pi/2)$ arbitrarily, and regard $\delta_{p,q}$ given by (4.5) as the constant depending only on p, q, and m; we thus write $\delta_{p,q} = \delta_{p,q}(m)$. By virtue of Lemma 4.3, we obtain not only the generation but the following $(L^p \times L^q) - (L^r \times L^s)$ estimates of the semigroup.

PROPOSITION 4.4. *For each* $\{p, q\}$ *satisfying* (4.1), *assume* (4.6). *Then* $-L$ *generates a bounded analytic semigroup* $\{e^{-tL}\}_{t\ge 0}$ *of class* (C_0) *on* $L^p_\sigma(\Omega) \times L^q(\Omega)$. *Furthermore, for* $j = 0, 1$ *and* $\{r, s\}$ *satisfying* (4.7), *there is a constant* $C = C(p, q, r, s; \alpha, \sigma, R; m)$ *such that*

$$\left\| \nabla^j e^{-tL} \begin{bmatrix} v \\ \theta \end{bmatrix} \right\|_{\{r,s\}} \le C\{t^{-(3/2)(1/p-1/r)-j/2} + t^{-(3/2)(1/q-1/s)-j/2}\} \left\| \begin{bmatrix} v \\ \theta \end{bmatrix} \right\|_{\{p,q\}}, \qquad (4.12;\ j)$$

for all $t > 0$ *and* $\{v, \theta\} \in L^p_\sigma(\Omega) \times L^q(\Omega)$.

Proof. By (4.8; 0) for $\{r, s\} = \{p, q\}$, the first part of this proposition follows from the standard theory of analytic semigroups as given for instance in Tanabe [19]. To obtain (4.12; j), we have only to evaluate the Dunford integral representation

$$\nabla^j e^{-tL} = \frac{1}{2\pi i} \int_\Gamma \nabla^j (\lambda + L)^{-1} e^{t\lambda}\, d\lambda \qquad (t > 0),$$

with use of (4.8; j). Here, the resolvent is integrated from $\infty e^{-i\varphi}$ to $\infty e^{i\varphi}$ along the path $\Gamma : \lambda = |\lambda| e^{\pm i\varphi}$ for an arbitrarily fixed $\varphi \in (\pi/2, \pi - \omega)$ provided that both $\beta + j/2$ and $\gamma + j/2$ are positive. When $\beta + j/2 = 0$ (that is, $r = p$ and $j = 0$), we split the estimate of the Dunford integral into two parts and replace Γ by $\Gamma_t : \lambda = |\lambda| e^{\pm i\varphi}$ ($|\lambda| \ge 1/t$), $\lambda = (1/t) e^{i \arg \lambda}$ ($|\arg \lambda| \le \varphi$) in the first integral:

$$\|e^{-tL}\|_{\{p,q\}\to\{p,s\}} \le \frac{C}{2\pi} \int_{\Gamma_t} |\lambda|^{-1} e^{t \operatorname{Re} \lambda} |d\lambda| + \frac{C}{2\pi} \int_\Gamma |\lambda|^{-1+\gamma} e^{t \operatorname{Re} \lambda} |d\lambda|.$$

The case where $\gamma + j/2 = 0$ is similarly treated. An elementary calculation thus implies the desired estimates (4.12; j). ∎

It should be noted that there is a positive number μ satisfying

$$\operatorname{Re} \sigma(L_{p,q}) > \mu, \qquad (4.13)$$

which means $\sigma(L_{p,q}) \subset \{\lambda \in \mathbb{C};\ \mathrm{Re}\ \lambda > \mu\}$, on account of $(0 \in)\ \Sigma_\omega \subset \rho(-L_{p,q})$. Thus we will derive the exponential decay of e^{-tL} as $t \to \infty$.

PROPOSITION 4.5. *For each* $\{p, q\}$ *satisfying* (4.1), *assume* (4.6). *Suppose that* $\mu > 0$ *satisfies* (4.13). *Then there is a constant* $C_\mu = C_\mu(p, q; \alpha, \sigma, R; m)$ *such that*

$$\left\| e^{-tL} \begin{bmatrix} v \\ \theta \end{bmatrix} \right\|_{\{p,q\}} \leq C_\mu e^{-\mu t} \left\| \begin{bmatrix} v \\ \theta \end{bmatrix} \right\|_{\{p,q\}}, \tag{4.14}$$

for all $t \geq 0$ *and* $\{v, \theta\} \in L^p_\sigma(\Omega) \times L^q(\Omega)$.

By proposition 4.5 estimates (4.12; j) can be improved.

PROPOSITION 4.6. *For each* $\{p, q\}$ *satisfying* (4.1), *assume* (4.6). *Suppose that* $\mu > 0$ *satisfies* (4.13). *Then, for* $j = 0, 1$ *and* $\{r, s\}$ *satisfying* (4.7), *there is a constant* $C_\mu = C_\mu(p, q, r, s; \alpha, \sigma, R; m)$ *such that*

$$\left\| \nabla^j e^{-tL} \begin{bmatrix} v \\ \theta \end{bmatrix} \right\|_{\{r,s\}} \leq C_\mu \{ t^{-(3/2)(1/p-1/r)-j/2} + t^{-(3/2)(1/q-1/s)-j/2} \} e^{-\mu t} \left\| \begin{bmatrix} v \\ \theta \end{bmatrix} \right\|_{\{p,q\}}, \tag{4.15; j}$$

for all $t > 0$ *and* $\{v, \theta\} \in L^p_\sigma(\Omega) \times L^q(\Omega)$.

The following obvious lemma is used in the proof of the propositions above.

LEMMA 4.7. *For any* $\mu > 0$ *satisfying* (4.13), *there is a number* $\mu^* > \mu$ *such that* μ^* *also satisfies* (4.13), *namely*, $\mathrm{Re}\ \sigma(L_{p,q}) > \mu^*$.

Proof. This is evident because $\sigma(L_{p,q})$ is a closed set. ∎

Proof of Proposition 4.5. By Lemma 4.7 combined with $\Sigma_\omega \subset \rho(-\mathrm{L}_{p,q})$, there is a number $\mu^* > \mu$ such that

$$H^{\mu^*}_\omega \equiv \Sigma_\omega \cup \{\lambda \in \mathbb{C};\ \mathrm{Re}\ \lambda \geq -\mu^*\} \subset \rho(-L_{p,q}).$$

Also, we can take $\varphi = \varphi(\mu^*) \in (\pi/2, \pi)$ so that

$$\Gamma \equiv \{\lambda \in \mathbb{C};\ \lambda = -\mu + |\lambda + \mu| e^{\pm i\varphi}\} \subset H^{\mu^*}_\omega.$$

In view of (4.8; 0), we have

$$\|(\lambda + L)^{-1}\|_{\{p,q\}\to\{p,q\}} \le C_{\mu^*} \qquad (\lambda \in H_\omega^{\mu^*}),$$

with a constant $C_{\mu^*} = C_{\mu^*}(p, q; \alpha, \sigma, R; m)$. Consequently,

$$\left\| e^{-tL}\begin{bmatrix} v \\ \theta \end{bmatrix}\right\|_{\{p,q\}} = \frac{1}{2\pi}\left\| \int_\Gamma (\lambda + L)^{-1} e^{t\lambda}\, d\lambda \begin{bmatrix} v \\ \theta \end{bmatrix}\right\|_{\{p,q\}}$$
$$\le \frac{C_{\mu^*} e^{-\mu t}}{\pi}\int_0^\infty e^{t\rho\cos\varphi}\, d\rho \cdot \left\|\begin{bmatrix} v \\ \theta \end{bmatrix}\right\|_{\{p,q\}}$$
$$= \frac{C_{\mu^*} e^{-\mu t}}{-\pi t\cos\varphi}\left\|\begin{bmatrix} v \\ \theta \end{bmatrix}\right\|_{\{p,q\}},$$

for $t > 0$. This combined with the boundedness of the semigroup (Proposition 4.4) implies (4.14). ∎

Proof of Proposition 4.6. For given $\mu > 0$, it is possible to take $\eta = \eta(\mu) > 0$ so small that $\mu/(1 - \eta)$ also satisfies (4.13) on account of Lemma 4.7. Proposition 4.5 then asserts that

$$\left\| e^{-tL}\begin{bmatrix} v \\ \theta \end{bmatrix}\right\|_{\{p,q\}} \le C_{\mu/(1-\eta)} e^{-(\mu/(1-\eta))t}\left\|\begin{bmatrix} v \\ \theta \end{bmatrix}\right\|_{\{p,q\}} \qquad (t \ge 0).$$

This combined with (4.12; j) leads us to

$$\left\| \nabla^j e^{-tL}\begin{bmatrix} v \\ \theta \end{bmatrix}\right\|_{\{r,s\}} = \left\| \nabla^j e^{-\eta tL} e^{-(1-\eta)tL}\begin{bmatrix} v \\ \theta \end{bmatrix}\right\|_{\{r,s\}}$$
$$\le C\{(\eta t)^{-(3/2)(1/p-1/r)-j/2} + (\eta t)^{-(3/2)(1/q-1/s)-j/2}\}\left\| e^{-(1-\eta)tL}\begin{bmatrix} v \\ \theta \end{bmatrix}\right\|_{\{p,q\}}$$
$$\le C_\mu\{t^{-(3/2)(1/p-1/r)-j/2} + t^{-(3/2)(1/q-1/s)-j/2}\}e^{-\mu t}\left\|\begin{bmatrix} v \\ \theta \end{bmatrix}\right\|_{\{p,q\}},$$

which proves (4.15; j). ∎

Remark 4.1. In view of (4.9) and (4.10), L^{-1} is compact since L_0^{-1} is so. Thus $\sigma(L_{p,q})$ consists of eigenvalues each with finite multiplicity. It is not known, however, whether the eigenvalue whose real part attains the minimum of Re $\sigma(L_{p,q})$ is simple or not.

5. Global Existence and Exponential Decay

In the present section we will prove Theorem 2. Let $\{\bar{v}, \bar{\theta}\}$ be the steady solution of class (m, K_0R) in Theorem 1. In what follows we write $\{v, \theta\}$ (resp. $\{a, b\}$) in place of $\{v - \bar{v}, \theta - \bar{\theta}\}$ (resp. $\{a - \bar{v}, b - \bar{\theta}\}$). Then such a disturbance $\{v, \theta\}$ should satisfy (2.4)–(2.6), which is now rewritten as

$$\begin{gathered} \frac{d}{dt}\begin{bmatrix} v \\ \theta \end{bmatrix} + L\begin{bmatrix} v \\ \theta \end{bmatrix} + \begin{bmatrix} P(v \cdot \nabla v) \\ v \cdot \nabla \theta \end{bmatrix} = 0 \qquad (t > 0), \\ v(0) = a, \theta(0) = b, \end{gathered} \tag{CP}$$

in a suitable $L^p_\sigma(\Omega) \times L^q(\Omega)$, where L is the linearized operator studied in the previous section. Since $-L$ generates a bounded analytic semigroup (Proposition 4.4), the integral equation associated with (CP) is given by

$$\begin{bmatrix} v(t) \\ \theta(t) \end{bmatrix} = e^{-tL}\begin{bmatrix} a \\ b \end{bmatrix} - \int_0^t e^{-(t-s)L}\begin{bmatrix} P(v \cdot \nabla v) \\ v \cdot \nabla \theta \end{bmatrix}(s)\, ds. \tag{IP}$$

We may concentrate the analysis on the case $1 < m \le 3/2$. Indeed, in this case, it seems to be difficult to derive a decay estimate of $\|\theta(t)\|_\infty$ directly. To deduce such an estimate, we first prove the global existence result (with exponential decay) in the following form.

Proposition 5.1. *For given $m \in (1, 3/2]$, let p and r_0 satisfy*

$$m \le p < \frac{3m}{3-m} \qquad (\le 3), \tag{5.1}$$

$$3 < r_0 < \infty, \qquad \frac{1}{r_0} + \frac{1}{m} < 1, \tag{5.2}$$

respectively. Suppose that

$$\{a, b\} \in L^3_\sigma(\Omega) \times L^p(\Omega).$$

Let $\delta_{p,q}(m)$ be the constant given by (4.5) (we have fixed $\omega \in (0, \pi/2)$ arbitrarily), and set

$$\delta' = \delta'(p, r_0; m) \equiv \min\{\delta_{3r_0/(3+r_0), pr_0/(p+r_0)}(m), \delta_{3,p}(m)\}.$$

If

$$\left(\frac{1}{\sigma}+\sigma\right)\alpha < \delta', \qquad (1+\sigma)R < \delta',$$

then there exists a positive constant $\varepsilon' = \varepsilon'(\alpha, \sigma, R; \rho, r_0)$ *such that whenever*

$$\|a\|_3 + \|b\|_\rho < \varepsilon',$$

(CP) *has a unique strong* ($L^3 \times L^\rho$) *solution* $\{v, \theta\}$ *on* $[0, \infty)$. *Furthermore, if* $\mu > 0$ *satisfies* (4.13) *for* $\{p, q\} = \{3r_0/(3+r_0), \rho r_0/(\rho + r_0)\}$ *and* $\{3, \rho\}$, *then the estimates*

$$\begin{aligned} &\|v(t)\|_r \le C_\mu t^{-(3/2)(1/3-1/r)} e^{-\mu t}(\|a\|_3 + \|b\|_\rho) \qquad (3 \le r < \infty), \\ &\|\theta(t)\|_s \le C'_\mu t^{-(3/2)(1/\rho-1/s)} e^{-\mu t}(\|a\|_3 + \|b\|_\rho) \qquad \left(\rho \le s < \frac{3\rho}{3-\rho}\right) \end{aligned} \tag{5.3}$$

$$\begin{aligned} &\|\nabla v(t)\|_r \le C_\mu t^{-(3/2)(1/3-1/r)-1/2} e^{-\mu t}(\|a\|_3 + \|b\|_\rho) \qquad (3 \le r < r_0), \\ &\|\nabla \theta(t)\|_s \le C'_\mu t^{-(3/2)(1/\rho-1/s)-1/2} e^{-\mu t}(\|a\|_3 + \|b\|_\rho) \qquad \left(\rho \le s < \left(\frac{1}{\rho}+\frac{1}{r_0}-\frac{1}{3}\right)^{-1}\right), \end{aligned} \tag{5.4}$$

hold for $t > 0$ *with constants* $C_\mu = C_\mu(r; \alpha, \sigma, R; \rho, r_0)$, $C'_\mu = C'_\mu(s; \alpha, \sigma, R; \rho, r_0)$.

Remark 5.1. The constant $\delta'(\rho, r_0; m)$ in Proposition 5.1 is well defined since both $\{3r_0/(3+r_0), \rho r_0/(\rho + r_0)\}$ and $\{3, \rho\}$ satisfy (4.1), which is observed in the proofs of Lemmas 5.2 and 5.3 below.

To prove Proposition 5.1, we give two lemmas.

LEMMA 5.2. *Let* ρ *satisfy* (5.1) *and let*

$$\left(\frac{1}{\sigma}+\sigma\right)\alpha < \delta_{3,\rho}(m), \qquad (1+\sigma)R < \delta_{3,\rho}(m).$$

Suppose that $\mu > 0$ *satisfies* (4.13) *for* $\{p, q\} = \{3, \rho\}$. *If* $\{r, s\}$ *fulfills*

$$3 \le r < \infty, \tag{5.5}$$

$$\frac{1}{3}-\frac{1}{r}=\frac{1}{\rho}-\frac{1}{s}, \tag{5.6}$$

then

$$\left\| \nabla^j e^{-tL} \begin{bmatrix} v \\ \theta \end{bmatrix} \right\|_{\{r,s\}} \le C_\mu t^{-(3/2)(1/3-1/r)-j/2} e^{-\mu t} \left\| \begin{bmatrix} v \\ \theta \end{bmatrix} \right\|_{\{3,\rho\}}, \tag{5.7;j}$$

for $t > 0$, $j = 0, 1$, *and* $\{v, \theta\} \in L^3_\sigma(\Omega) \times L^\rho(\Omega)$ *with a constant* $C_\mu = C_\mu(r; \alpha, \sigma, R; \rho)$.

Proof. By (5.1) it is easily seen that $\{3, \rho\}$ satisfies (4.1); in fact,

$$-\frac{2}{3} < \frac{1}{3} - \frac{1}{m} \le \frac{1}{3} - \frac{1}{\rho} < \frac{2}{3},$$

$$\frac{1}{3} - \frac{1}{\rho} < \frac{2}{3} - \frac{1}{m} < 1 - \frac{1}{m}, \qquad \frac{1}{\rho} > \frac{1}{m} - \frac{1}{3}.$$

Moreover, (5.6) together with

$$0 \le \frac{3}{2}\left(\frac{1}{3} - \frac{1}{r}\right) < \frac{1}{2} \le 1 - \frac{j}{2} \qquad (j = 0, 1),$$

asserts that $\{r, s\}$ fulfills (4.7). This proves (5.7;j) by virtue of Proposition 4.6. ∎

LEMMA 5.3. *Let* ρ *and* r_0 *satisfy* (5.1) *and* (5.2), *respectively. Let*

$$\left(\frac{1}{\sigma} + \sigma\right)\alpha < \delta_{3r_0/(3+r_0),\rho r_0/(\rho+r_0)}(m), \qquad (1 + \sigma)R < \delta_{3r_0/(3+r_0),\rho r_0/(\rho+r_0)}(m).$$

Suppose that $\mu > 0$ *satisfies* (4.13) *for* $\{p, q\} = \{3r_0/(3 + r_0), \rho r_0/(\rho + r_0)\}$. *If* $\{r, s\}$ *satisfies not only* (5.6) *but*

$$0 \le 1 + 3\left(\frac{1}{r_0} - \frac{1}{r}\right) \le 2 - j, \tag{5.8;j}$$

for each $j = 0, 1$, *then*

$$\left\| \nabla^j e^{-tL} \begin{bmatrix} P(v \cdot \nabla w) \\ v \cdot \nabla\theta \end{bmatrix} \right\|_{\{r,s\}} \le C_\mu t^{-(3/2)(1/r_0-1/r)-(1+j)/2} e^{-\mu t} \|v\|_{r_0} (\|\nabla w\|_3 + \|\nabla\theta\|_\rho), \tag{5.9;j}$$

for $t > 0$, $v \in L^{r_0}_\sigma(\Omega)$, $\nabla w \in L^3(\Omega)$ *and* $\nabla\theta \in L^\rho(\Omega)$ *with a constant* $C_\mu = C_\mu(r; \alpha, \sigma, R; \rho, r_0)$.

Proof. Define $\{p, q\}$ by

$$\frac{1}{p}=\frac{1}{r_0}+\frac{1}{3}, \qquad \frac{1}{q}=\frac{1}{r_0}+\frac{1}{\rho}.$$

Then by (5.1) and (5.2), we have $1 < p, q < \infty$. Further, $\{p, q\}$ satisfies (4.1) since $\{3, \rho\}$ does (see the proof of Lemma 5.2). It is also observed that $\{r, s\}$ satisfying (5.6) and (5.8; j) fulfills (4.7) for such $\{p, q\}$; indeed,

$$\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)=\frac{3}{2}\left(\frac{1}{q}-\frac{1}{s}\right)=\frac{1}{2}\left\{1+3\left(\frac{1}{r_0}-\frac{1}{r}\right)\right\}\in\left[0, 1-\frac{j}{2}\right].$$

Thus (4.15; j) of Proposition 4.6 combined with the Hölder inequality implies (5.9; j). ∎

Remark 5.2. For the integrability of (5.9; j) at $t = 0$, the number r should satisfy $3(1/r_0 - 1/r) < 1 - j$.

Proof of Proposition 5.1. We set

$$\Phi\begin{bmatrix} v \\ \theta \end{bmatrix}(t) = \text{the right-hand side of (IP)},$$

and find a fixed point of Φ in the Banach space ($\mathscr{B}$ denotes the class of bounded continuous functions)

$$X=\left\{\{v,\theta\};\ \begin{array}{l} t^{\beta}e^{\mu t}\{v,\theta\}\in\mathscr{B}([0,\infty); L^{r_0}_{\sigma}(\Omega)\times L^{s_0}(\Omega)) \\ t^{1/2}e^{\mu t}\{\nabla v,\nabla\theta\}\in\mathscr{B}([0,\infty); L^{3}(\Omega)\times L^{\rho}(\Omega)) \\ \text{with values zero at } t=0 \end{array}\right\}$$

with norm

$$\left\|\begin{bmatrix} v \\ \theta \end{bmatrix}\right\|_X=\sup_{t>0} t^{\beta}e^{\mu t}(\|v(t)\|_{r_0}+\|\theta(t)\|_{s_0})+\sup_{t>0} t^{1/2}e^{\mu t}(\|\nabla v(t)\|_3+\|\nabla\theta(t)\|_{\rho}),$$

where

$$s_0=\left(\frac{1}{\rho}+\frac{1}{r_0}-\frac{1}{3}\right)^{-1}, \qquad \beta=\frac{3}{2}\left(\frac{1}{3}-\frac{1}{r_0}\right)=\frac{3}{2}\left(\frac{1}{\rho}-\frac{1}{s_0}\right)>0.$$

For $M > 0$, set

$$X_M = \left\{\{v, \theta\} \in X; \left\|\begin{bmatrix} v \\ \theta \end{bmatrix}\right\|_X \leq M\right\}.$$

Since $(1/\sigma + \sigma)\alpha$, $(1 + \sigma)R < \delta'$, we are able to apply Lemmas 5.2 and 5.3 for $j = 0, \{r, s\} = \{r_0, s_0\}$, and for $j = 1, \{r, s\} = \{3, \rho\}$. Then, by essentially the same argument as in the Navier–Stokes equations (Fujita and Kato [5], Giga and Miyakawa [10]), we can take a constant $\varepsilon' = \varepsilon'(\alpha, \sigma, R; \rho, r_0)$ such that whenever $\|a\|_3 + \|b\|_\rho < \varepsilon'$, the following hold true with a constant $M = C(\|a\|_3 + \|b\|_\rho)$:

$$\Phi\begin{bmatrix} v \\ \theta \end{bmatrix} \in X_M \qquad \text{for all} \begin{bmatrix} v \\ \theta \end{bmatrix} \in X_M,$$

Φ is a contraction mapping on X_M.

Thus (IP) has a solution $\{v, \theta\}$ in X_M, which satisfies (5.3) for $\{r, s\} = \{r_0, s_0\}$ and (5.4) for $\{r, s\} = \{3, \rho\}$. The proof of uniqueness is standard, so it is omitted. By evaluating (IP) with use of Lemmas 5.2 and 5.3 again (see also Remark 5.2), we are easily led to (5.3) and (5.4). Following the argument of [5, 10, 11], we can see the local Hölder continuity with respect to t of the nonlinear term in $L^3_\sigma(\Omega) \times L^\rho(\Omega)$, to show that $\{v, \theta\}$ is actually a strong $(L^3 \times L^\rho)$ solution to (CP). ▮

We are now in a position to prove Theorem 2 by making use of Proposition 5.1 twice.

Proof of Theorem 2. We take and fix $r_0 = r_0(m)$ satisfying (5.2). We first appeal to Proposition 5.1 for $\rho = m$: if $(1/\sigma + \sigma)\alpha$, $(1 + \sigma)R < \delta'(m, r_0; m)$, then there exists a constant $\varepsilon = \varepsilon(\alpha, \sigma, R; m) \equiv \varepsilon'(\alpha, \sigma, R; m, r_0(m))$ such that whenever

$$\|a\|_3 + \|b\|_m < \varepsilon(\alpha, \sigma, R; m),$$

(CP) has a unique strong $(L^3 \times L^m)$ solution $\{v, \theta\}$ on $[0, \infty)$ with the decay estimates (5.3)–(5.4) for $\rho = m$, where $\mu > 0$ is an arbitrarily fixed number satisfying (4.13) for $\{p, q\} = \{3r_0/(3 + r_0), mr_0/(m + r_0)\}$ and $\{3, m\}$. Since the estimate $\|\nabla v(t)\|_r$ for some $r > 3$ has been obtained, it follows from the Gagliardo–Nirenberg inequality that

$$\|v(t)\|_\infty = o(e^{-\mu t}),$$

as $t \to \infty$. On the other hand, we cannot directly derive the decay estimate of $\|\theta(t)\|_\infty$ because $(1/m + 1/r_0 - 1/3)^{-1} < 3$ in case $m \leq 3/2$.

To deduce the desired decay, we will employ Proposition 5.1 once more. By (5.2) it is possible to choose and fix $\rho_0 = \rho_0(m)$ so that

$$m < \frac{3r_0}{2r_0 - 3} < \rho_0 < \frac{3m}{3 - m}. \tag{5.10}$$

By (5.3) (for $\rho = m$) with $\{r, s\} = \{3, \rho_0\}$, we can also take $T > 0$ so large that

$$\|v(T)\|_3 + \|\theta(T)\|_{\rho_0} < \varepsilon'(\alpha, \sigma, R; \rho_0, r_0),$$

under the condition $(1/\sigma + \sigma)\alpha$, $(1 + \sigma)R < \delta'(\rho_0, r_0; m)$. Set

$$\delta = \delta(m) \equiv \min\{\delta'(m, r_0(m); m), \delta'(\rho_0(m), r_0(m); m)\},$$

and suppose that $(1/\sigma + \sigma)\alpha$, $(1 + \sigma)R < \delta(m)$. Then, regarding $\{v(T), \theta(T)\} \in L^3_\sigma(\Omega) \times L^{\rho_0}(\Omega)$ as initial data and employing Proposition 5.1 for $\rho = \rho_0$, we can deduce (5.4) (for $\rho = \rho_0$), where $\mu > 0$ is an arbitrarily fixed number satisfying (4.13) for $\{p, q\} = \{3r_0/(3 + r_0), \rho_0 r_0/(\rho_0 + r_0)\}$ and $\{3, \rho_0\}$. So we have the decay estimate of $\|\nabla\theta(t)\|_s$ for some $s > 3$ because (5. 10) implies

$$\left(\frac{1}{\rho_0} + \frac{1}{r_0} - \frac{1}{3}\right)^{-1} > 3,$$

and because we have the uniqueness of solutions. We thus obtain

$$\|\theta(t)\|_\infty = o(e^{-\mu t}),$$

as $t \to \infty$. ∎

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